Notes for AA214, Chapter 11

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NUMERICAL DISSIPATION

1. The governing equations of most physical systems are dominated by convective and dissipative processes.

2. In processes governed by nonlinear equations, such as the Euler and Navier-Stokes equations, there can be a continual production of high-frequency components of the solution, leading, for example, to the formation of shock waves.

3. In a real physical problem, the production of high frequencies is eventually limited by viscosity.

4. However, when we solve the Euler equations numerically, we have neglected viscous effects.
5. Thus the numerical approximation must contain some inherent dissipation to limit the production of high-frequency modes.
6. Although numerical approximations to the Navier-Stokes equations contain dissipation through the viscous terms, this can be insufficient, especially at high Reynolds numbers, due to the limited grid resolution which is practical.
7. Unless the relevant length scales are resolved, some form of added numerical dissipation is required

8. The addition of numerical dissipation is tantamount to intentionally introducing nonphysical behavior, and must be carefully controlled such that the error introduced is not excessive.

9. A centered approximation to a first derivative is non-dissipative, i.e., the eigenvalues of the associated circulant matrix (with periodic boundary conditions) are pure imaginary.

10. A non-centered (upwind) approximation to a first derivative is dissipative, e.g., 1^{st} Order backward differencing leads to eigenvalues which have a real part.

11. For our model wave equation, the sign of the wave speed a combined with a specific choice of difference operator can lead a positive real part of the eigenvalues and therefore inherent instability.

One-Sided First-Derivative Differencing

1. Starting with our favorite wave equation

$$\frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x} \tag{1}$$

2. Consider the generalize three point difference operator

$$\begin{aligned} -a(\delta_x u)_j &= \frac{-a}{2\Delta x}[-(1+\beta)u_{j-1} + 2\beta u_j + (1-\beta)u_{j+1}] \\ &= \frac{-a}{2\Delta x}[(-u_{j-1} + u_{j+1}) + \beta(-u_{j-1} + 2u_j - u_{j+1})] \end{aligned} \tag{2}$$

Backward/Forward Difference Operator

1. The operator is divided into
   (a) Antisymmetric component $(-u_{j-1} + u_{j+1})/2\Delta x$
   (b) Symmetric component $\beta(-u_{j-1} + 2u_j - u_{j+1})/2\Delta x$
   (c) Antisymmetric component: 2^{nd} centered difference.
   (d) With $\beta \neq 0$, the operator is only 1^{st} accurate.
   (e) A *backward* difference operator is given by $\beta = 1$
   (f) A *forward* difference operator is given by $\beta = -1$.

Type Dependent Differencing

1. For periodic boundary conditions, matrix operator is

$$-a\delta_x = \frac{-a}{2\Delta x} B_p(-1-\beta, 2\beta, 1-\beta)$$

2. The eigenvalues of this matrix are,
   $m = 0, 1, \ldots, M-1$

$$\lambda_m = \frac{-a}{\Delta x}\left\{\beta\left[1-\cos\left(\frac{2\pi m}{M}\right)\right] + i\sin\left(\frac{2\pi m}{M}\right)\right\}$$

3. For a positive, forward difference ($\beta = -1$):
   $\Re(\lambda_m) > 0$

4. Centered difference operator ($\beta = 0$): $\Re(\lambda_m) = 0$
5. For a positive, backward difference: $\Re(\lambda_m) < 0$.
6. The forward difference operator is inherently unstable
7. Centered/backward operators are inherently stable.
8. If a is negative, the roles are reversed.
9. $\Re(\lambda_m) \neq 0$, the solution will either grow or decay.
10. Equation 2 can be used with a switching scheme:
    (a) If $a > 0$, set $\beta = 1$
    (b) If $a < 0$, set $\beta = -1$

The Modified Partial Differential Equation

1. Taylor series expansion of the terms in Eq. 2.

$$(\delta_x u)_j = \frac{1}{2\Delta x}\left[\, 2\Delta x \left(\frac{\partial u}{\partial x}\right)_j - \beta \Delta x^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_j + \frac{\Delta x^3}{3}\left(\frac{\partial^3 u}{\partial x^3}\right)_j - \frac{\beta \Delta x^4}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_j + \dots \right] \tag{3}$$

2. The antisymmetric portion introduces odd derivative terms.

3. The symmetric portion introduces even derivatives.

The Modified Partial Differential Equation

1. Substituting into Eq. 1 gives

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \frac{a\beta\Delta x}{2}\frac{\partial^2 u}{\partial x^2} - \frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + \frac{a\beta\Delta x^3}{24}\frac{\partial^4 u}{\partial x^4} + \ldots \tag{4}$$

2. The *modified PDE* we are really solving.
3. *Consistent* with Eq. 1, two equations identical when $\Delta x \to 0$.
4. In practice, Δx can be small, but it is *not* zero
5. Each term given by Eq. 4 is excited to some degree.
6. The actual PDE being solved is different than the original, Eq.1

Effect of Errors Terms: Modified PDE

1. Consider the simple linear partial differential equation

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^3 u}{\partial x^3} + \tau\frac{\partial^4 u}{\partial x^4} \tag{5}$$

2. Periodic BC and impose an IC: $u = e^{i\kappa x}$.

3. Wave-like solution to Eq. 5 of the form
   $u(x,t) = e^{i\kappa x}e^{(r+is)t}$

4. r and s satisfy the condition

$$r + is = -ia\kappa - \nu\kappa^2 - i\gamma\kappa^3 + \tau\kappa^4$$

or

$$r = -\kappa^2(\nu - \tau\kappa^2), \quad s = -\kappa(a + \gamma\kappa^2)$$

5. Solution contains both amplitude and phase terms.

$$u = \underbrace{e^{-\kappa^2(\nu-\tau\kappa^2)}}_{\text{amplitude}} \; \underbrace{e^{i\kappa[x-(a+\gamma\kappa^2)t]}}_{\text{phase}} \tag{6}$$

6. The amplitude of the solution depends only upon ν and τ, the coefficients of the even derivatives in Eq. 5

7. Phase depends only on a and γ, the coefficients of the odd derivatives.

Effect of Errors Terms: Modified PDE

1. Wave speed a is positive
   (a) Backward difference ($\beta = 1$) modified PDE:
   $\nu - \tau\kappa^2 > 0$
   (b) Amplitude of the solution decays.
   (c) *Deliberately adding* dissipation to the PDE.
   (d) Forward difference scheme ($\beta = -1$) is equivalent to *deliberately adding* a destabilizing term to the PDE.

2. Phase of the solution in Eq. 6

   (a) Speed of propagation is $a + \gamma\kappa^2$

   (b) Modified PDE, Eq. 4,$\gamma = -a\Delta x^2/6$.

   (c) Phase speed of the numerical solution is *less* than the actual phase speed, *dispersion*.

Artificial Dissipation

1. Note that the use of one-sided differencing schemes is not the only way to introduce dissipation.
2. Any symmetric component in the spatial operator introduces dissipation (or amplification).
3. Therefore, one could choose $\beta = 1/2$ in Eq. 2.
4. The resulting spatial operator is not one-sided, but it is dissipative.
5. Biased schemes use more information on one side of the node than the other.

Third-Order backward Difference

1. Third-order backward-biased scheme is given by

$$\begin{aligned}(\delta_x u)_j &= \frac{1}{6\Delta x}(u_{j-2} - 6u_{j-1} + 3u_j + 2u_{j+1}) \\ &= \frac{1}{12\Delta x}[(u_{j-2} - 8u_{j-1} + 8u_{j+1} - u_{j+2}) \\ &\quad +(u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2})] \qquad (7)\end{aligned}$$

2. The antisymmetric component of this operator is the fourth-order centered difference operator.

3. The symmetric component approximates $\Delta x^3 u_{xxxx}/12$.

4. Operator produces fourth-order accuracy in phase with a third-order dissipative term.

The Lax-Wendroff Method

1. Previous discussion implies:
   (a) Introduce numerical dissipation using one-sided differencing
   (b) Backward differencing if the wave speed is positive
   (c) Forward differencing if the wave speed is negative.
2. Lax-Wendroff Method: introduces dissipation independent of the sign of the wave speed
3. Differs conceptually from the methods considered previously

Derivation of Lax-Wendroff

1. Taylor-series expansion in time:

$$u(x, t+h) = u + h\frac{\partial u}{\partial t} + \frac{1}{2}h^2\frac{\partial^2 u}{\partial t^2} + O(h^3) \tag{8}$$

2. Replace time derivatives with space derivatives using PDE

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}, \qquad \frac{\partial^2 u}{\partial t^2} = -a\frac{\partial \frac{\partial u}{\partial t}}{\partial x} = a^2\frac{\partial^2 u}{\partial x^2} \tag{9}$$

3. Replace the space derivatives 3-point centered difference

$$u_j^{(n+1)} = u_j^{(n)} - \frac{1}{2}\frac{ah}{\Delta x}(u_{j+1}^{(n)} - u_{j-1}^{(n)})$$
$$+\frac{1}{2}\left(\frac{ah}{\Delta x}\right)^2 (u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)}) \tag{10}$$

Stability of Lax-Wendroff

1. For periodic boundary conditions, fully-discrete matrix operator:

$$\vec{u}_{n+1} = B_p\left(\frac{1}{2}\left[\frac{ah}{\Delta x} + \left(\frac{ah}{\Delta x}\right)^2\right], 1 - \left(\frac{ah}{\Delta x}\right)^2, \frac{1}{2}\left[-\frac{ah}{\Delta x} + \left(\frac{ah}{\Delta x}\right)^2\right]\right)\vec{u}_n$$

2. Eigenvalues of this matrix are, $m = 0, 1, \ldots, M-1$

$$\sigma_m = 1 - \left(\frac{ah}{\Delta x}\right)^2\left[1 - \cos\left(\frac{2\pi m}{M}\right)\right] - i\frac{ah}{\Delta x}\sin\left(\frac{2\pi m}{M}\right) \quad (11)$$

3. For $|\frac{ah}{\Delta x}| \leq 1$: eigenvalues have modulus less than or equal to unity

4. Method is stable *independent of the sign of* a.

5. $CFL = |\frac{a\,h}{\Delta x}|$: Courant (or CFL) number.

6. Ratio of the distance traveled by a wave in one time step to the mesh spacing.

Modified PDE for Lax-Wendroff

1. See text for derivation of Modified PDE

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} =$$
$$-\frac{a}{6}(\Delta x^2 - a^2 h^2)\frac{\partial^3 u}{\partial x^3} - \frac{a^2 h}{8}(\Delta x^2 - a^2 h^2)\frac{\partial^4 u}{\partial x^4} + \ldots$$

2. Leading error terms appear on the right side of the equation.
3. Odd derivatives on the right side lead to unwanted dispersion
4. Even derivatives lead to dissipation,or amplification,

depending on the sign.

5. Leading error term in the Lax-Wendroff method is dispersive and proportional to

$$-\frac{a}{6}(\Delta x^2 - a^2h^2)\frac{\partial^3 u}{\partial x^3} = -\frac{a\Delta x^2}{6}(1 - C_n^2)\frac{\partial^3 u}{\partial x^3}$$

6. Dissipative term is proportional to

$$-\frac{a^2h}{8}(\Delta x^2 - a^2h^2)\frac{\partial^4 u}{\partial x^4} = -\frac{a^2h\Delta x^2}{8}(1 - C_n^2)\frac{\partial^4 u}{\partial x^4}$$

7. Term has the appropriate sign and hence the scheme is truly dissipative as long as $C_n \leq 1$.

MacCormack's method

1. MacCormack's Method is closely related to Lax-Wendroff
2. MacCormack's Time-marching method, (see Chapter 6 of text)

$$
\begin{aligned}
\tilde{u}_{n+1} &= u_n + h u'_n \\
u_{n+1} &= \frac{1}{2}[u_n + \tilde{u}_{n+1} + h\tilde{u}'_{n+1}]
\end{aligned}
$$

3. Use first-order backward differencing in the first stage
4. Use first-order forward differencing in the second stage,

5. Dissipative second-order method is obtained.
6. For the linear convection equation

$$\begin{aligned} \tilde{u}_j^{(n+1)} &= u_j^{(n)} - \frac{ah}{\Delta x}(u_j^{(n)} - u_{j-1}^{(n)}) \\ u_j^{(n+1)} &= \frac{1}{2}[u_j^{(n)} + \tilde{u}_j^{(n+1)} - \frac{ah}{\Delta x}(\tilde{u}_{j+1}^{(n+1)} - \tilde{u}_j^{(n+1)})] \end{aligned}$$

7. Can be shown to be identical to the Lax-Wendroff method.
8. MacCormack's method has the same dissipative and dispersive properties as the Lax-Wendroff method.
9. The two methods differ when applied to nonlinear hyperbolic systems

UPWIND SCHEMES

1. Numerical dissipation can be introduced in the spatial difference operator using one-sided difference schemes.

2. Based on stability arguements: the direction of the one-sided operator depends on the sign of the wave speed.

3. *Hyperbolic system* of equations: wave speeds can be both positive and negative.

4. In the wave equation example:
   (a) If $a > 0$, Backward differencing
   (b) If $a < 0$, Forward differencing
5. Eigenvalues of the flux Jacobian for the one-dimensional Euler equations
   (a) $u, u + c, u - c$ where c is the speed of sound.
   (b) When the flow is subsonic $u < c$, these are of mixed sign.
   (c) To apply one-sided differencing schemes to such systems, some form of splitting is required.

Characteristic Splitting

1. Consider again the linear convection equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{12}$$

2. With the sign of a arbitrary .

3. Rewrite Eq. 12

$$\frac{\partial u}{\partial t} + (a^{+} + a^{-})\frac{\partial u}{\partial x} = 0 \quad ; \quad a^{\pm} = \frac{a \pm |a|}{2}$$

   (a) If $a \geq 0$, then $a^{+} = a \geq 0,\ \ a^{-} = 0.$

   (b) If $a \leq 0$, then $a^{+} = 0,\ \ a^{-} = a \leq 0.$

(c) For the a^+ (≥ 0) term we can safely backward difference.

(d) For the a^- (≤ 0) term forward difference.

4. Basic concept behind upwind methods

5. Some decomposition or splitting of the fluxes into terms which have positive and negative characteristic speeds so that appropriate differencing schemes can be chosen.

Flux-Vector Splitting

1. Linear, constant-coefficient, hyperbolic system of PDE

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0 \tag{13}$$

   (a) Can be decoupled into characteristic equations

$$\frac{\partial w_i}{\partial t} + \lambda_i \frac{\partial w_i}{\partial x} = 0 \tag{14}$$

   (b) Wave speeds, λ_i: eigenvalues of the Jacobian matrix, A

(c) The w_i's are the characteristic variables.

(d) Backward difference if the wave speed, λ_i, is positive,

(e) Forward difference if the wave speed is negative.

$\pm$ Characteristic Splitting

1. In gerneral, we do not go to characteristic space (w_i), but stay in the flux space u, A, f
2. Split the matrix of eigenvalues, Λ, into two components

$$\Lambda = \Lambda^+ + \Lambda^- \tag{15}$$

$$\Lambda^+ = \frac{\Lambda + |\Lambda|}{2}, \qquad \Lambda^- = \frac{\Lambda - |\Lambda|}{2} \tag{16}$$

3. Λ^+ contains the positive eigenvalues
4. Λ^- contains the negative eigenvalues

$\pm$ Type Dependent Differencing

1. Rewrite the system in terms of characteristic variables as

$$\frac{\partial w}{\partial t} + \Lambda \frac{\partial w}{\partial x} = \frac{\partial w}{\partial t} + \Lambda^{+} \frac{\partial w}{\partial x} + \Lambda^{-} \frac{\partial w}{\partial x} = 0 \qquad (17)$$

2. Spatial terms split into two components according to the sign of the wave speeds.

3. Backward differencing for the $\Lambda^{+} \frac{\partial w}{\partial x}$ term

4. Forward differencing for the $\Lambda^{-} \frac{\partial w}{\partial x}$ term.

$\pm$ Flux vector Splitting

1. Premultiplying by X, the matrix of right eigenvectors of A, and inserting the product $X^{-1}X$ in the spatial terms gives

$$\frac{\partial Xw}{\partial t} + \frac{\partial X\Lambda^+ X^{-1} Xw}{\partial x} + \frac{\partial X\Lambda^- X^{-1} Xw}{\partial x} = 0 \qquad (18)$$

2. Define

$$A^+ = X\Lambda^+ X^{-1}, \qquad A^- = X\Lambda^- X^{-1} \qquad (19)$$

   (a) A^+ has all positive eigenvalues, by construction

   (b) A^- has all negative eigenvalues, by construction

3. Recall that $u = Xw$

$$\frac{\partial u}{\partial t} + \frac{\partial A^{+} u}{\partial x} + \frac{\partial A^{-} u}{\partial x} = 0 \tag{20}$$

4. Finally the split flux vectors are defined as

$$f^{+} = A^{+} u, \qquad f^{-} = A^{-} u \tag{21}$$

5. Leading to the Flux Vector Splitting Form

$$\frac{\partial u}{\partial t} + \frac{\partial f^{+}}{\partial x} + \frac{\partial f^{-}}{\partial x} = 0 \tag{22}$$

Flux vector Splitting

1. In the linear case, the definition of the split fluxes follows directly from the definition of the flux, $f = Au$.
2. For the Euler equations, f is also equal to Au as a result of their homogeneous property, as discussed in Appendix C of the text.

3. Note that

$$f = f^+ + f^- \tag{23}$$

4. Thus by applying backward differences to the f^+ term and forward differences to the f^- term, we are in effect solving the characteristic equations in the desired manner.

Implicit Implementation of FVS

1. Implicit time-marching: need Jacobians of the split flux vectors.

2. In the nonlinear case,

$$\frac{\partial f^+}{\partial u} \neq A^+, \qquad \frac{\partial f^-}{\partial u} \neq A^- \tag{24}$$

3. Signs of the Jacobians must have corresponding $\pm$ eigenvalues

$$A^{++} = \frac{\partial f^+}{\partial u}, \qquad A^{--} = \frac{\partial f^-}{\partial u} \tag{25}$$

4. For the Euler equations:
   (a) A^{++} has eigenvalues which are all positive
   (b) A^{--} has all negative eigenvalues.

Artificial Dissipation Concepts

1. Numerical dissipation can be introduced by using one-sided differencing schemes together with some form of flux splitting.

2. Dissipation can also be introduced by adding a symmetric component to an antisymmetric (dissipation-free) operator.

3. Generalize the concept of upwinding to include any scheme in which the symmetric portion of the operator is dissipative.

Construction: Artificial Dissipation

1. Define

$$\left(\delta_x^a u\right)_j = \frac{u_{j+1} - u_{j-1}}{2\Delta x}, \qquad \left(\delta_x^s u\right)_j = \frac{-u_{j+1} + 2u_j - u_{j-1}}{2\Delta x}$$

2. Applying $\delta_x = \delta_x^a + \delta_x^s$ to the spatial derivative in Eq. 14 is stable if $\lambda_i \geq 0$ and unstable if $\lambda_i < 0$.

3. $\delta_x = \delta_x^a - \delta_x^s$ is stable if $\lambda_i \leq 0$ and unstable if $\lambda_i > 0$.

4. Appropriate implementation is thus

$$\lambda_i \delta_x = \lambda_i \delta_x^a + |\lambda_i| \delta_x^s$$

5. Extension to a hyperbolic system by applying the

above approach to the characteristic variables

$$\delta_x(Au) = \delta_x^a(Au) + \delta_x^s(|A|u)$$

$$\delta_x f = \delta_x^a f + \delta_x^s(|A|u)$$

$$|A| = X|\Lambda|X^{-1}$$

6. The second spatial term is known as *artificial dissipation.*
7. Sometimes referred to as artificial diffusion or artificial viscosity.
8. Appropriate choices of δ_x^a and δ_x^s, this approach can be related to the upwind approach.

9. It is common to use the following operator for δ_x^s

$$(\delta_x^s u)_j = \frac{\epsilon}{\Delta x}(u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2})$$

10. ϵ is a problem-dependent coefficient.

11. Symmetric operator approximates $\epsilon \Delta x^3 u_{xxxx}$ and thus introduces a third-order dissipative term.

12. Appropriate value of ϵ, this often provides sufficient damping of high frequency modes without greatly affecting the low frequency modes.

Nonlinear Artificial Dissipation, JST

1. 2^{nd} and 4^{th} derivative AD employing a pressure gradient switch and spectral radius scaling.

$$\nabla_x \left(\sigma_{j+1} + \sigma_j\right) \left(\epsilon_j^{(2)} \Delta_x Q_j - \epsilon_j^{(4)} \Delta_x \nabla_x \Delta_x Q_j\right) / \Delta x$$

   with $\epsilon_j^{(2)} = \epsilon_2 \max(\Upsilon_{j+1}, \Upsilon_j, \Upsilon_{j-1})$,
   $\Upsilon_j = \frac{|p_{j+1} - 2p_j + p_{j-1}|}{|p_{j+1} + 2p_j + p_{j-1}|}, \epsilon_j^{(4)} = \max(0, \epsilon_4 - \epsilon_j^{(2)})$

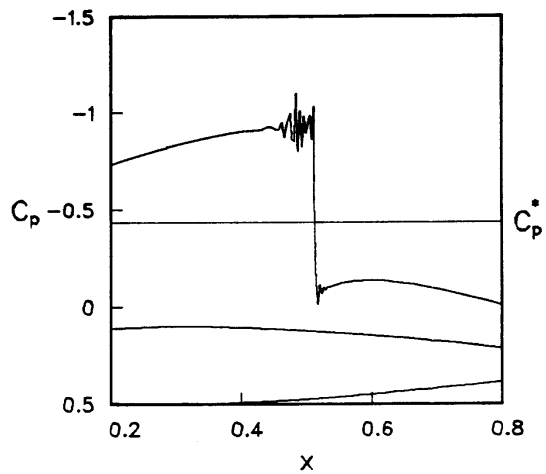
2. Typical values of the constants are $\epsilon_2 = 1/4$ and $\epsilon_4 = 1/100$.

3. The term σ_j is a spectral radius scaling and is defined as $\sigma_j = |u| + a$ with a the speed of sound.

Linear Constant Coefficient AD

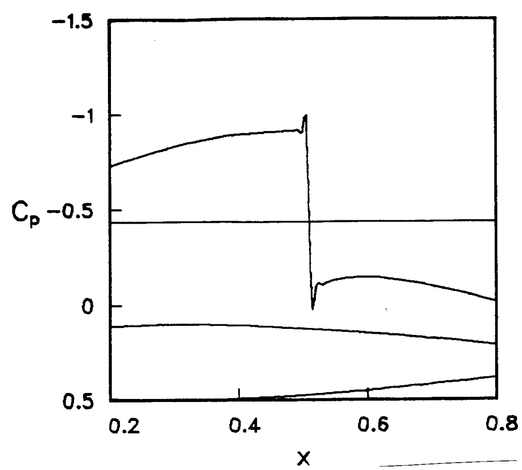
1. Early forms of Artificial Dissipation were linear ($\sigma = 1$), without the pressure switch.

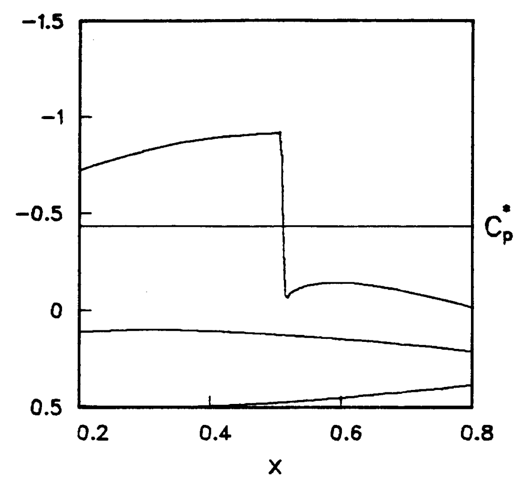
2. NACA0012, transonic solution



Non-Linear Artificial Dissipation

1. Current





4^{th} Difference Only $\qquad 2^{nd} - 4^{th}$